Arnold's proof of the nonexistence of a solution to the quintic equation

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Square roots

Here is a proof that $\sqrt{2}$ is not rational.

Suppose to the contrary that $\sqrt{2} = \frac{p}{q}$ without any common factors.

Then $\sqrt{2} = \frac{2q-p}{p-q}$ but with a smaller denominator leading to a contradiction.

For $n \geqslant 3$, $\sqrt[n]{2}$ is not rational either. If not, as before, we must have

$$p^n = 2q^n = q^n + q^n$$

for a pair of integer p and q. But this contradicts the Fermat's last theorem!

$\sqrt{2}$ and $-\sqrt{2}$ can't be algebraically distinguished, that is, if $\sqrt{2}$ is the solution of a polynomial equation with rational coefficients, then so is $-\sqrt{2}$ and vice-versa. Such pairs are called conjugate.

More generally, two real numbers a and b are conjugate over $\mathbb{Q}$ if for all polynomials p with coefficients in $\mathbb{Q}$,

$$p(a) = 0 \iff p(b) = 0.$$

Similarly, two complex numbers z, z' are said to be conjugate if for all polynomials with coefficients in $\mathbb{R}$

$$p(z) = 0 \iff p(z') = 0.$$

The two numbers i and $-i$ are indistinguishable.

Definition: Let $k \geq 0$, and let $\left(z_1, \ldots, z_k\right)$, $\left(z_1', \ldots, z_k'\right)$ be $k-$tuples of complex numbers. Then $\left(z_1, \ldots, z_k\right)$ and $\left(z_1', \ldots, z_k'\right)$ are conjugate over $\mathbb{Q}$ if for all polynomials p over $\mathbb{Q}$ in k variables

$$p\left(z_1, \ldots, z_k\right) = 0 \iff p\left(z_1', \ldots, z_k'\right) = 0.$$

The symmetry group of a polynomial: Write $(s_1, \ldots, s_k)$ for its distinct solutions in $\mathbb{C}$. The Galois group of p is

$\mathrm{Gal}(p) = \left\{\sigma \in S_k : \left(s_1, \ldots, s_k\right) \textit{ and } \left(s_{\sigma(1)}, \ldots, s_{\sigma(k)}\right)\right.$ are conjugate$\}$

'Distinct solutions' means that we ignore any repetition of roots: if $p(t) = t^5(t-1)^9$, then $k = 2$ and $\left\{s_1, s_2\right\} = \{0,1\}$.

Informally, let us say that a complex number is radical if it can be obtained from the rationals using only the usual arithmetic operations and kth roots. For example, $\dfrac{\frac{1}{2} + \sqrt[3]{\sqrt[5]{2} - \sqrt[2]{7}}}{\sqrt[4]{6 + \sqrt[3]{\frac{2}{3}}}}$ is radical, whichever square root, cube root, etc., we choose. A polynomial over $\mathbb{Q}$ is solvable (or soluble) by radicals if all of its complex roots are radical.

Every quadratic over $\mathbb{Q}$ is solvable by radicals. This follows from the quadratic formula: $\frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$ is visibly a radical number.

Theorem of Galois

What determines if a polynomial is solvable by radicals? The amazing answer to this question was given by Galois.

Theorem: Suppose that p is a polynomial over $\mathbb{Q}$. Then p is solvable by radicals if and only if the Galois group $\mathrm{Gal}(p)$ is solvable.

We are going to however, discuss an elementary (by no means, trivial) proof due to Arnold.

Solution of polynomial equations

Let $p(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$ be a polynomial with complex coefficients $c_{n-1}, \ldots, c_0$. By the fundamental theorem of algebra, there are exactly n solutions to the equation $p(z) = 0$, say, $\{s_1, \ldots, s_n\}$. What happens to the solutions $\{s_1, \ldots, s_n\}$ if we move the coefficients $c_{n-1}, \ldots, c_0$ a little and what happens the other way around?

The answer involves permutations, loops, roots (of complex numbers), finally commutators.

It is clear that given a set of complex numbers $S = \{s_1, \ldots, s_n\}$, the set of solutions of

$$p(z) = 0, \text{ where } p(z) = (z - s_1)\cdots(z - s_n),$$

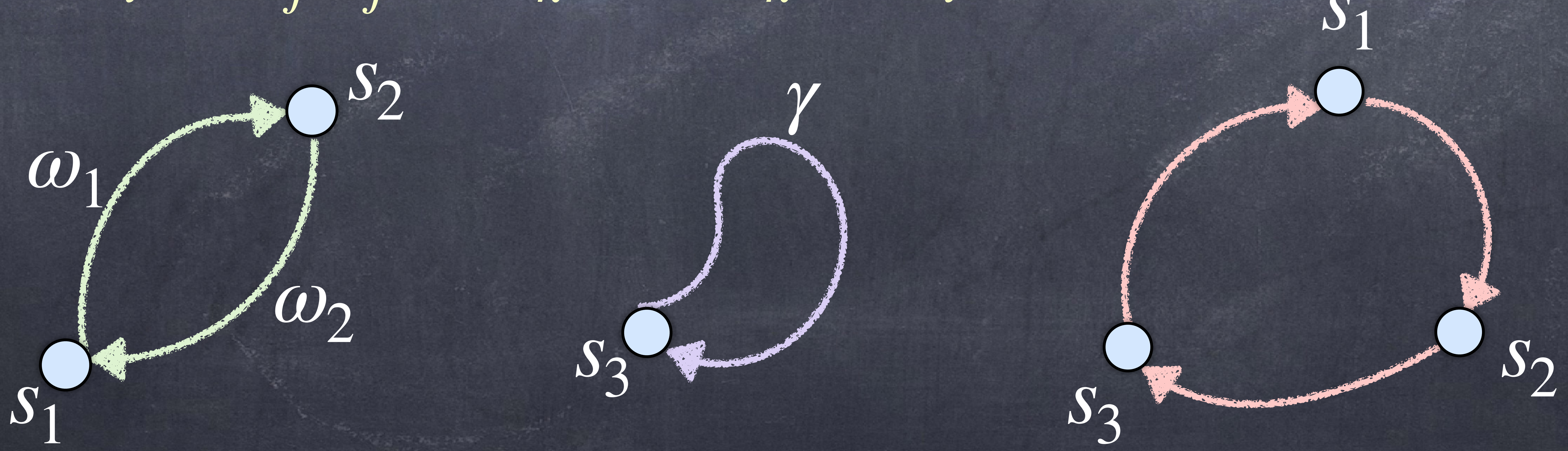
is exactly S. It is going the other way round, that is, how to find the solutions of a polynomial equation is not obvious.

Two kinds of permutations

We discuss two kinds of permutations, namely, transpositions and cycle:

- transpositions, denoted (ij), exchanging the position of two solutions, i.e., $s_i \to s_j$.

- cycles, denoted (ijk), exchanging the position of three solutions cyclically, i.e., $s_i \to s_j$, $s_j \to s_k$, and $s_k \to s_i$.



Loops and permutations

Locating the solutions $(s_1, \ldots, s_n)$ in $\mathbb{C}$, we can think of a permutation to be a path traveling from one solution to another.

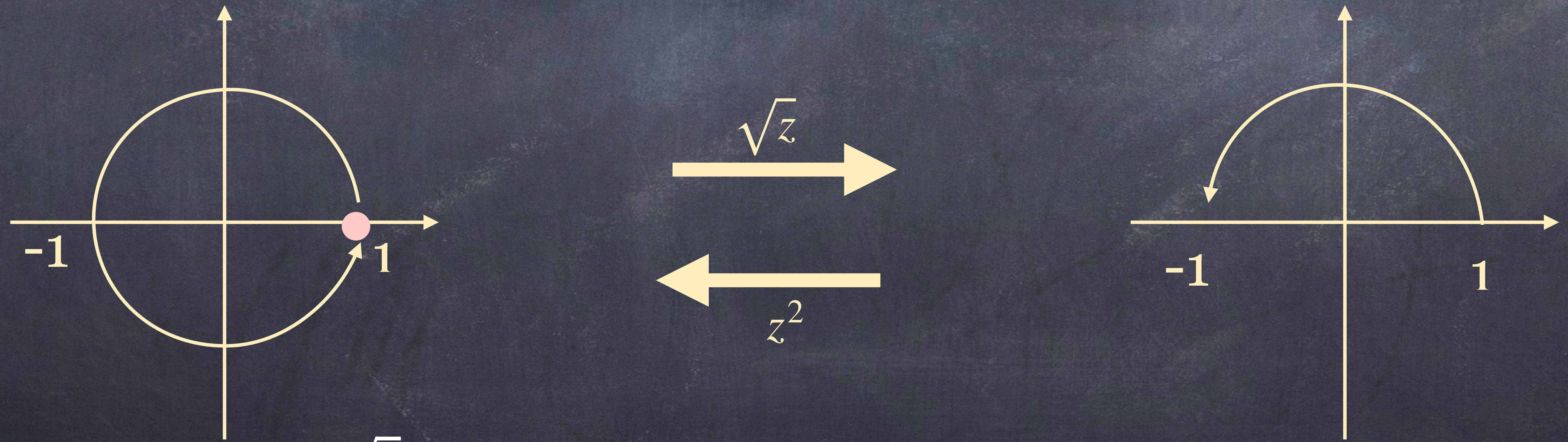
Paths in the complex plane are just continuous curves that connect two points (we assume that they do not self-intersect, otherwise things get unnecessarily complicated).

A path that closes, i.e., connects a point to itself, is called a loop and denoted γ.

These paths will be represented by arrows in all the figures, and will be used to induce permutations on the solutions $(s_1, \ldots, s_n)$.

How complex roots move around in $\mathbb{C}$

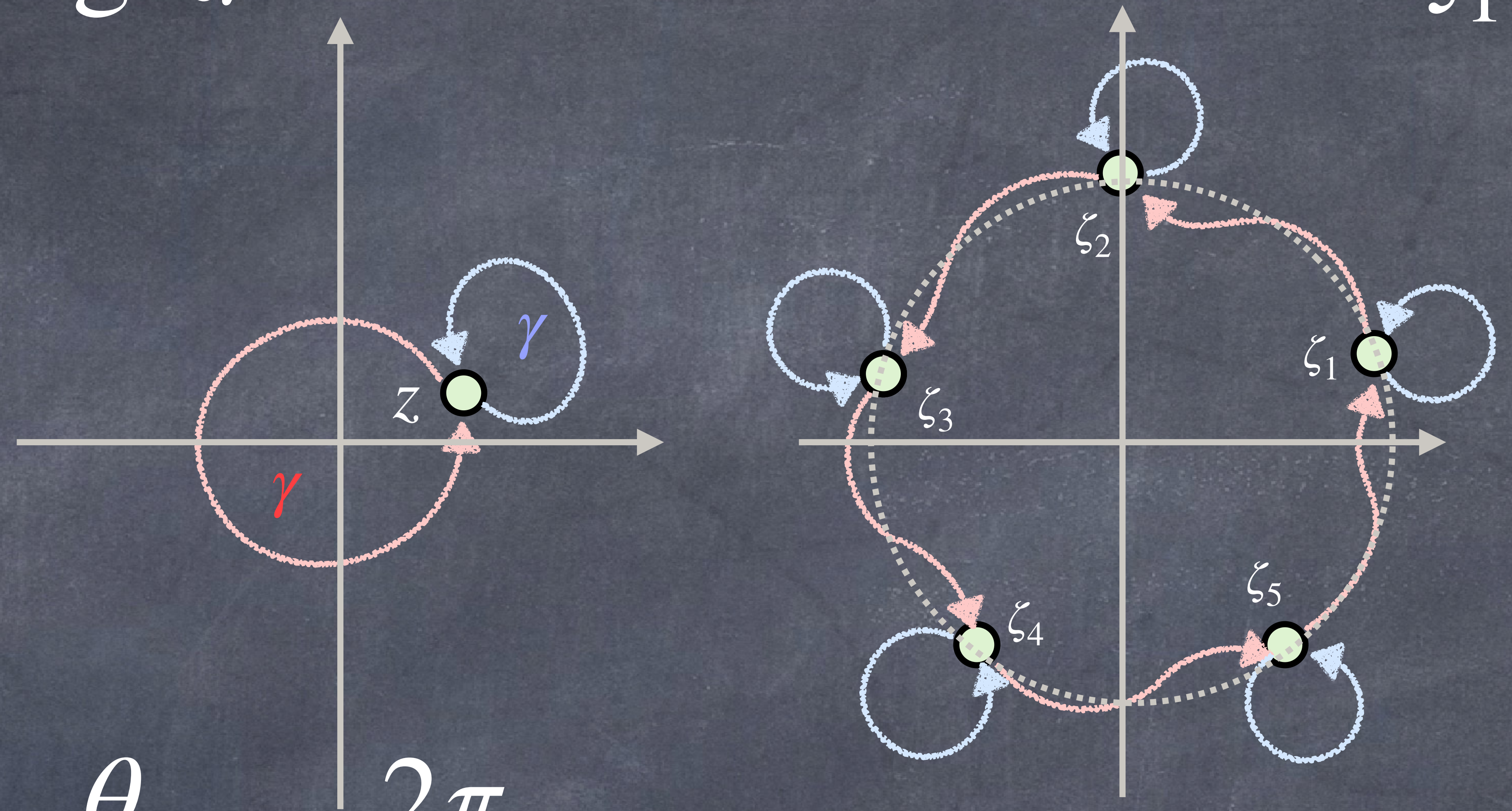
Fixing some complex number z, a root of z is some number ζ in $\mathbb{C}$ such that $\zeta^k = z$for some $k \in \mathbb{N}$. By the fundamental theorem of algebra, there are exactly k such kth root ζ of z; and z. Thus, $\sqrt[k]{z}$ denotes a multivalued function of the complex variable z. With a little abuse of notation, we let $\sqrt[k]{z}$ also denote any of the kth roots of z. Fixing $k \in \mathbb{N}$ and assuming that z itself follows a loop γ, we ask what kind of path $\sqrt[k]{z}$ follows. Notice that with $k = 2$, we have



When z follows a loop γ, $\sqrt{z}$ does not always follow a loop.

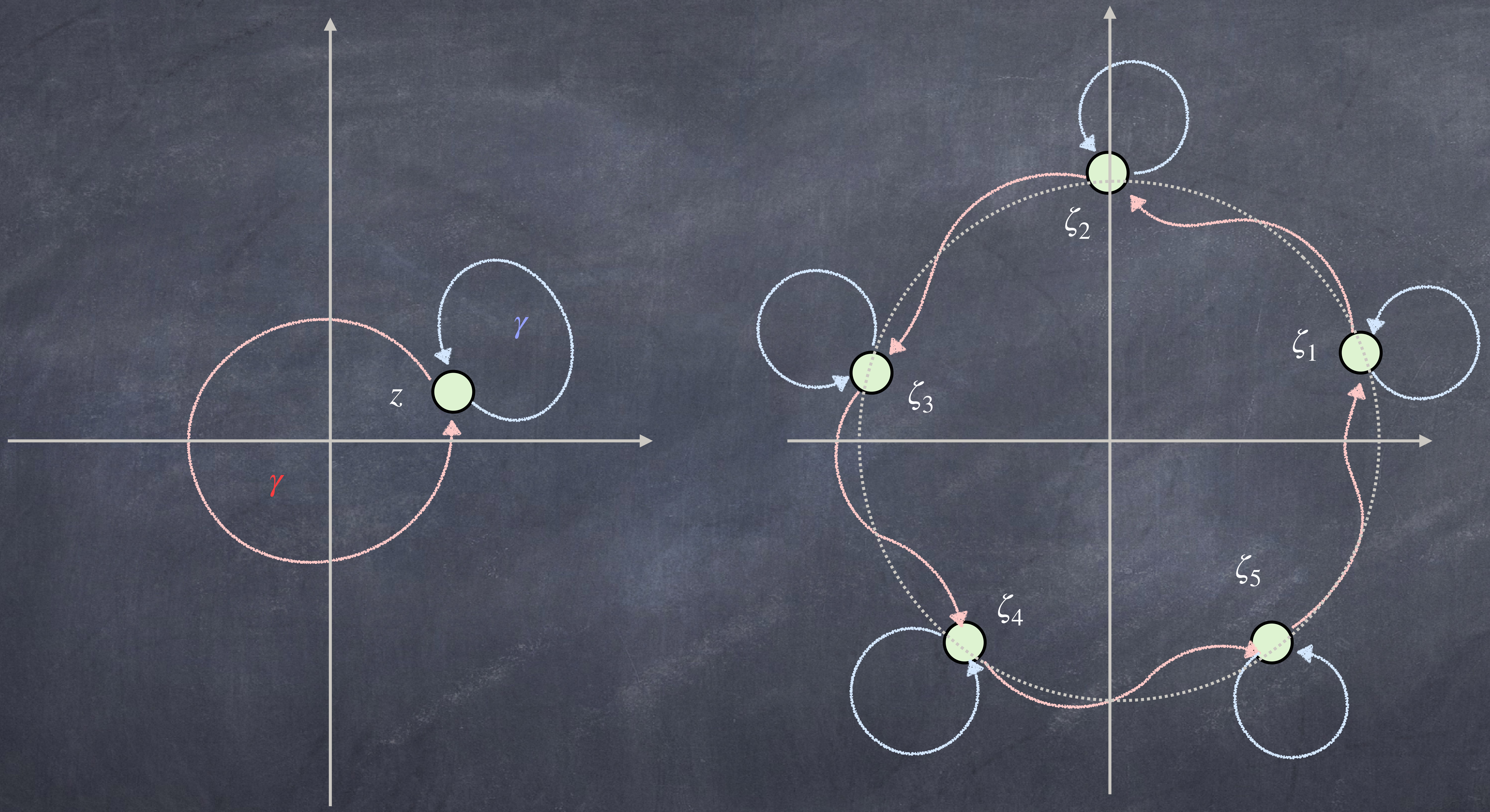
Set $z = re^{i\theta}$ with $r = |z|$ and $\theta = \arg z$, and write the kth roots $\zeta_1, \ldots, \zeta_k$ explicitly as

$$\zeta_\ell = \sqrt[k]{r}e^{i(\theta+2\ell\pi)/k}, \ \ell \in \{1,\ldots,k\}.$$



Thus, the argument $\arg(\zeta_\ell)$ of $\zeta_\ell = \dfrac{\theta}{k} + \ell\dfrac{2\pi}{k}$. This means that all roots are equally spaced on the circle of radius $\sqrt[k]{r}$, at angle $\dfrac{2\pi}{k}$ apart.

As z travels along a path γ, winding once around 0, its kth roots also move around since the $\arg(z)$ has gone from θ to $\theta + 2\pi$. Each kth root ζ_ℓ has moved to its closest, counter clock-wise neighbour $\zeta_{\ell+1}$. In particular, the roots have not completed a loop.



z
γ
γ
ζ_1
ζ_2
ζ_3
ζ_4
ζ_5

A formula for a solution s of a polynomial equation of the form $p(z) = 0$, in general, is of the form $s = R(c_0, c_1, \ldots, c_{n-1})$, where R is some function of the coefficients $c_0, \ldots, c_{n-1}$ of p obtained by using $+, -, \times, \div, \sqrt{\ }$.

A hierarchy of functions: The first ones, say R_0,that are made out of the coefficients $c_0, \ldots, c_{n-1}$ using only $+, -, \times, \div$. These are polynomial, or more generally, rational functions of the coefficients of the polynomial p.

Therefore, if two or more of these coefficients follow a loop the function of type R_0 also follows a loop.

This last property of R_0 functions is not shared by R_1 functions obtained from R_0functions by taking roots, as we have seen.

When $(c_0, \ldots, c_{n-1})$follow a loop, R_1-functions do not necessarily follow a loop.

We can build R_2-functions by taking roots of R_1-functions building higher order of nesting in the coefficients at each stage. Consider for example:

$R_0 = -\frac{c_3}{6}+c_0,$ or $c_2^3 + c_1,$

$R_1 = \sqrt{c_5^2 - 3}+\frac{1}{2}c_4^2 - \sqrt[3]{c_0},$

$R_2 = \sqrt[3]{\frac{2}{3}c_3^2 - c_1} + \sqrt{\frac{1}{3}c_2 + \sqrt[5]{c_5^2 + c_0 - 1}} + c_4, \ldots$

Quadratic Equation

First observation: Coefficients $c_0, c_1, \ldots, c_{n-1}$ are symmetric functions of the solutions $\{s_1, \ldots, s_n\}$. This follows since the polynomial $(z - s_1)\cdots(z - s_n)$ is independent of the ordering of the solutions $\{s_1, \ldots, s_n\}$.

For $n = 2$, if the two solutions s_1, s_2 are permuted using the transposition , the coefficients (c_0, c_1) each move on some path but they must come back to the original position when s_0 and s_1 exchange their position.

Theorem: There is no map $R_0 : \mathbb{C}^2 \to \mathbb{C}$ such that $R_0(c_0, c_1)$ is always a solution to the quadratic equation $p(z) = 0$, where $p(z) = z^2 + c_1 z + c_0$.

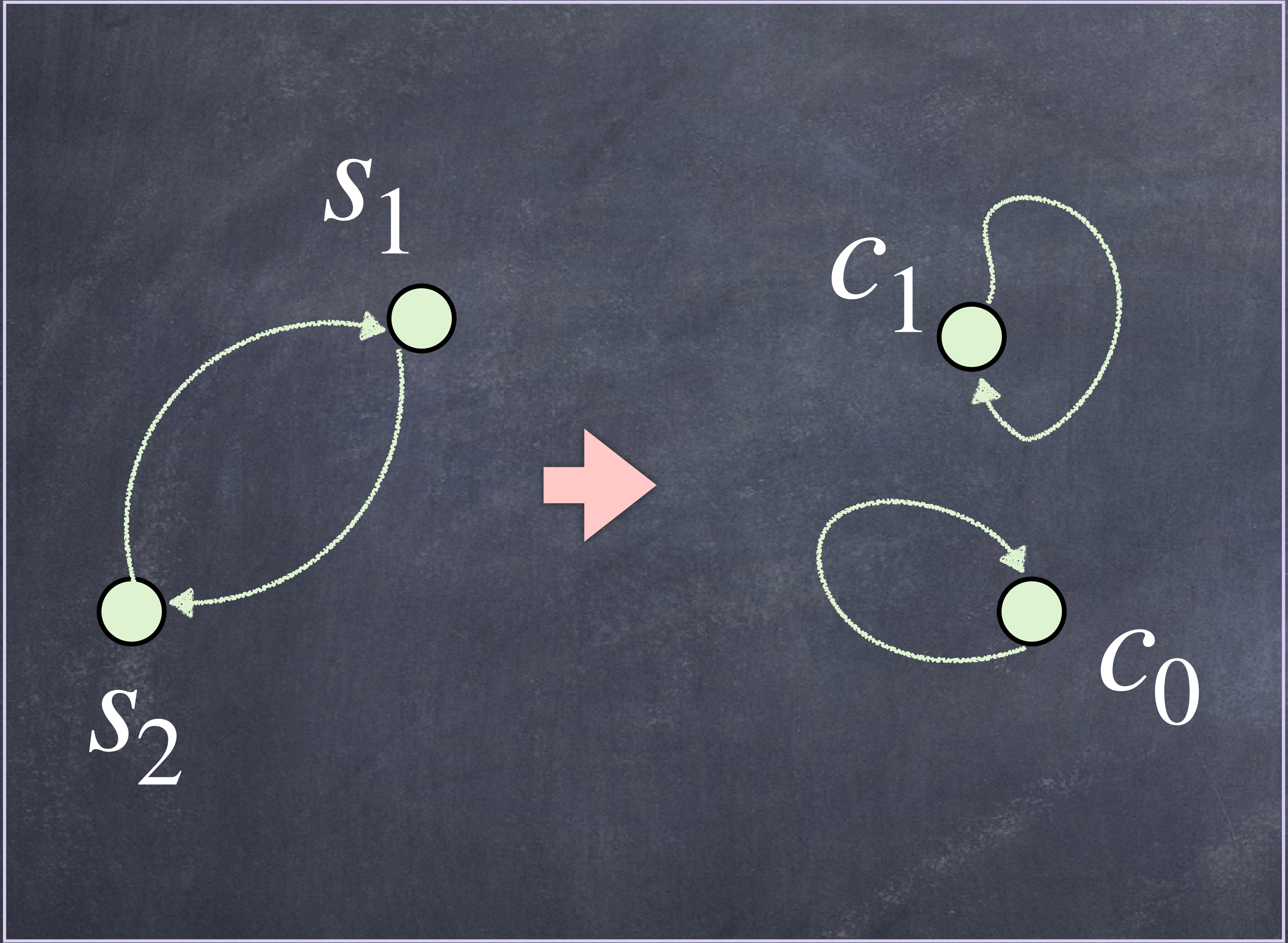
- The transposition (12) swaps the two solutions s_1 and s_2. Pick a continuous path $s_1(t)$ starting at $s_1 := s_1(0)$ and ending at $s_1(1) = s_2 = s_2(0)$. Also, choose a path s_2 starting at $s_2 = s_2(0)$ and ending at $s_2(1) = s_1 = s_1(0)$.

- The coefficients $c_0(t), c_1(t)$ are continuous symmetric functions of the solutions $\{s_1(t), s_2(t)\}$, therefore their final positions are the same as the initial positions. Thus, each c_0, c_1 defines a loop. The functions

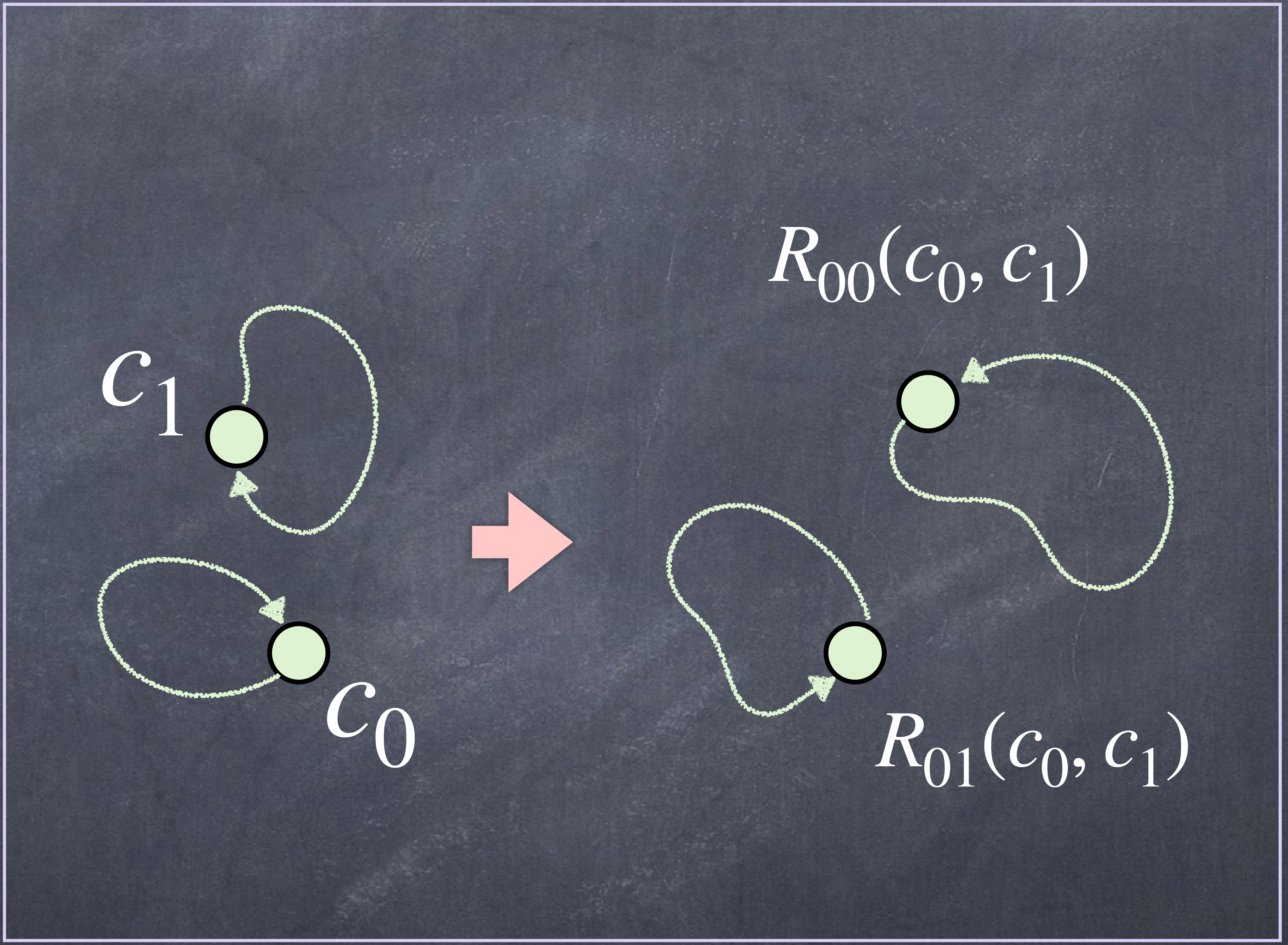
$$R_{0i}(c_0(t), c_1(t)) = s_i(t), \ i = 1,2,$$

being a continuous function of c_0, c_1, by hypothesis, will also follow its own loop.

- Consequently, as t runs from 0 to 1, the solutions s_1 and s_2 swap their positions while $R_{01}(c_0(0), c_1(0))$ and $R_{02}(c_0(1), c_1(1))$ coincide leading to a contradiction.



s_1
s_2
c_1
c_0



c_1
c_0
$R_{00}(c_0, c_1)$
$R_{01}(c_0, c_1)$

The cubic Equation

- Let $p(z) = 0$, where $p(z) = z^3 + c_2z^2 + c_1z + c_0$, be the cubic equation.

- Again, assume that we have solutions of the form

$$s_i = R_{1i}(c_0, c_1, c_2), i = 1,2,3,$$

  involving one level of radicals.

- We still have that each of the coefficients follow a loop as solutions permute.

- However, functions like $R_1$with radicals in them no longer follow a loop.

- We need a new idea!

Commutators

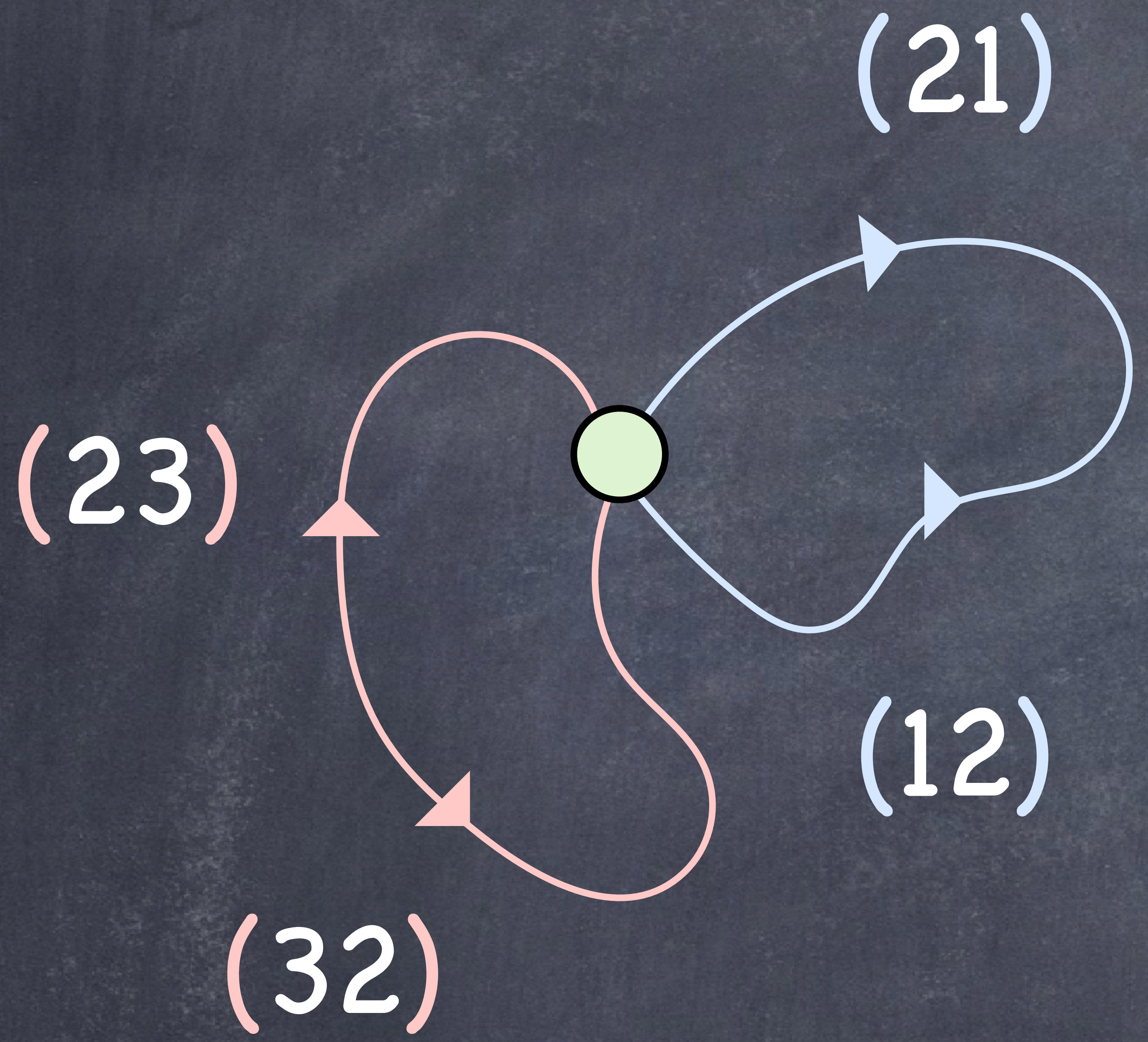
- Consider the transposition (12) that induces a loop $\gamma_1$on R_0 and an unclosed path ω_1 on R_1. Consider also (23), inducing a loop γ_2 on R_0 and a path ω_2 on R_1. Now perform the following sequence of transpositions, called the commutator of (12) and (23), and denoted

$$[(12),(23)] = (12)(23)(12)^{-1}(23)^{-1}.$$

- Since $(12)^{-1}$ is (21), and $(23)^{-1} = (32)$, it follows that $[(12),(23)]$ is the cycle (123). Indeed, this is true of any pair of transposition, namely, $[(ij),(jk)] = (ijk)$.

- Therefore, $[(12),(23)]$ permutes the three solutions $\left(s_1, s_2, s_3\right)$.

- Now, R_0follows a sequence of loops $\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$, which is itself a loop, however, R_1 follows a sequence of unclosed paths $\omega_1\omega_2\omega_1^{-1}\omega_2^{-1}$ (visiting other roots) but closes on itself by construction.



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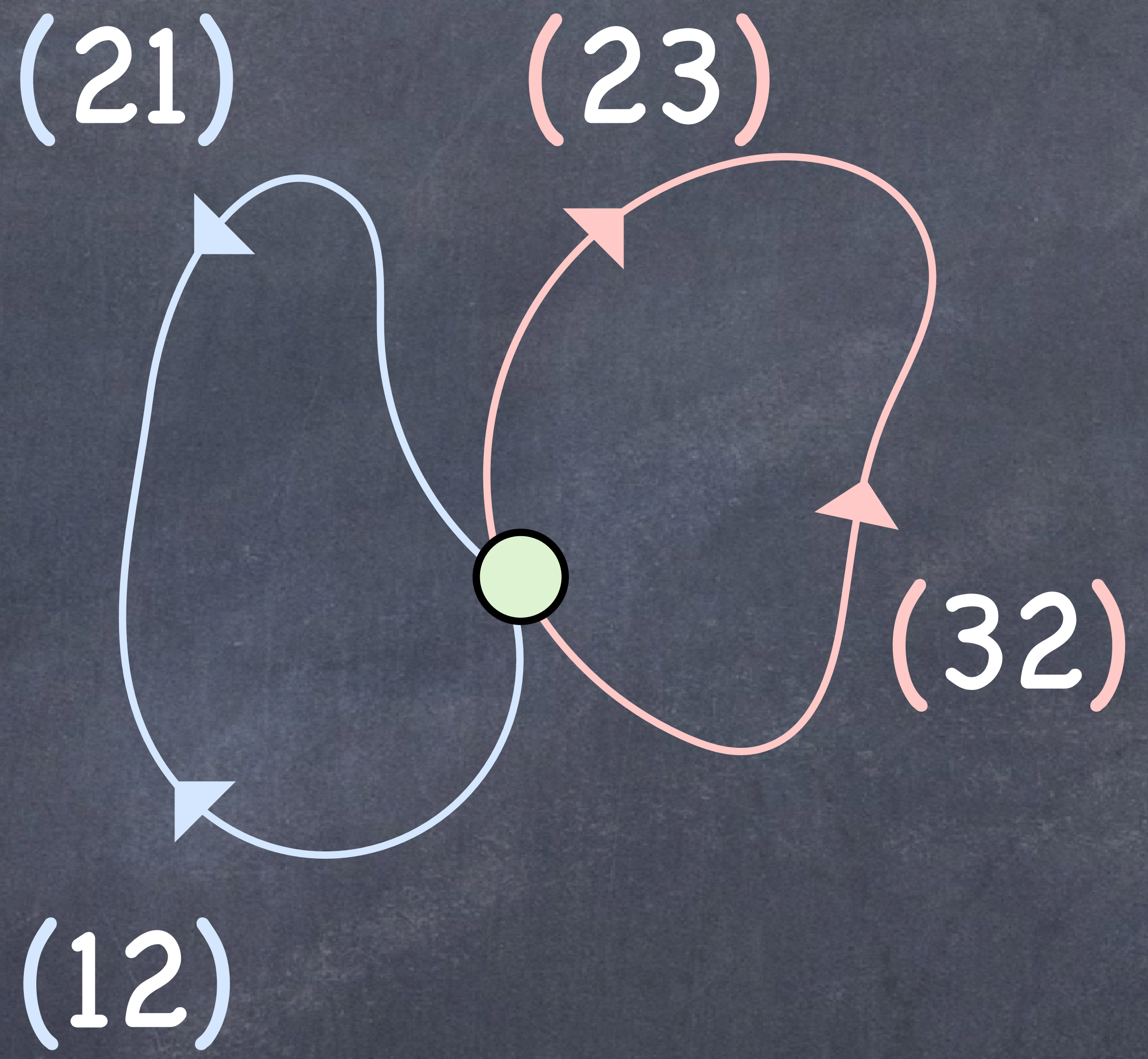
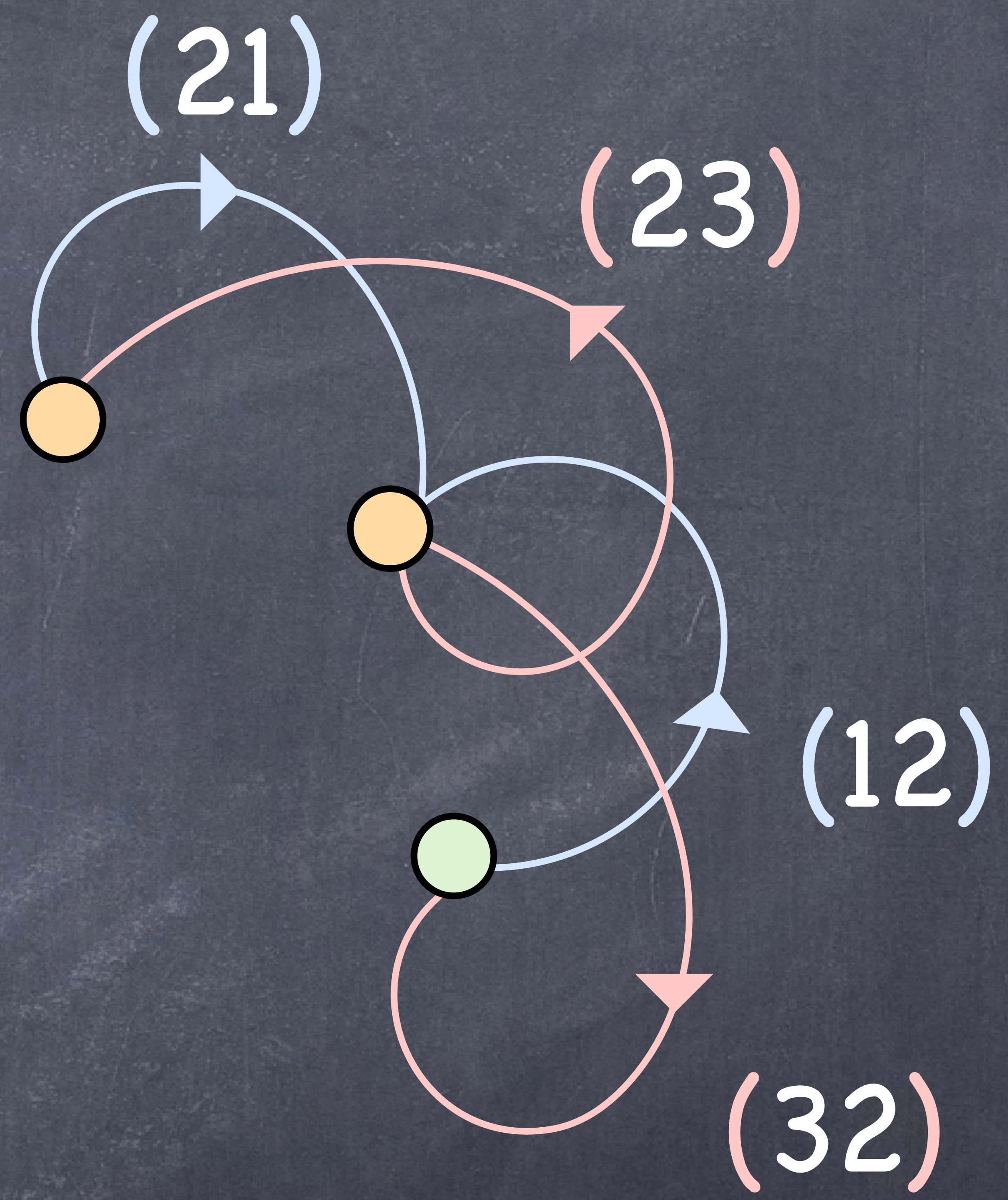


R_0



R_1

- Suppose that (s_1, s_2, s_3) undergoes the permutation (123).
- Then both R_0 and R_1 follow a loop. Consequently, we can't have equalities:

$$s_i = R_{1i}(c_0, c_1, c_2), i = 1,2,3.$$

- Theorem: There is no map $R_1 : \mathbb{C}^3 \to \mathbb{C}$ such that $R_1(c_0, c_1, c_2)$ is always a solution to the cubic equation

$$p(z) = 0, \text{ where } p(z) = z^3 + c_2 z^2 + c_1 z + c_0.$$

The Quartic

- We have seen that solutions of a cubic equation, in general, cannot be written using functions of type R_1 (one level of roots).

- Now, for the quartic equation,

$$p(z) = 0, \text{ where } p(z) = z^4 + c_3 z^3 + c_2 z^2 + c_! z + c_0,$$

- Assume that we have a solution of the form:

$$s_i = R_{2i}(c_0, c_1, c_2, c_3), i = 1,2,3,4,$$

with two levels of the nesting of roots.

The proof consists of constructing an appropriate permutation of the solutions $\{s_1, s_2, s_3, s_4\}$.

- As before, like the method for the quadratic did not work for the cubic, the method for the cubic doesn't really work for the quartic.

- Hunt for a new idea again, this time, we look at a commutator of two cycles (123) and (234) and note that it indeed permutes the four solutions since $[(123),(2,3,4)] = (14)(23)$.

- Applying $(123) = [(12),(23)]$ followed by $(234) = [(23),(34)]$ to functions of type R_1 produces two closed loops γ_1 followed by γ_2 coming back to the original position.

- However, functions of type R_2 will move along two generally unclosed paths ω_1 and ω_2.

- Second, we apply these two paths backwards, in reverse, that is, $(432) = [(43), (32)]$ and then $(321) = [(32), (21)]$. During these two, R_1 -functions will follow $\gamma_2^{-1}\gamma_1^{-1}$, i.e. the previous loops backwards. Similarly, R_2-functions will travel along $\omega_2^{-1}\omega_1^{-1}$.

- Thus, the R_1-functions follow the loop $\gamma = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$; and R_2functions a sequence of unclosed paths $\omega_1\omega_2\omega_1^{-1}\omega_2^{-1}$, which closes on itself by construction.

- Our conclusion has therefore been reached: while $\left(s_1, s_2, s_3, s_4\right)$ undergoes the permutation $(14)(23)$ written as a commutator of commutators, any R_2 -function follows a loop.

The quintic

- Let $p(z) = 0$, where $p(z) = z^5 + c_4 z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0$ be the quintic equation. Suppose that

$$s_\mathrm{i} = R_{3i}\left(c_0, \ldots, c_4\right) \text{ for } i \in \{1,\ldots,5\},$$

where the functions R_{3i} has three nested levels of roots.

- Following what is done for $n = 2,3,4$, note that (1) all R_k-functions with $k = 0,1,2$, will follow a loop from a commutator of commutators of the solutions (as in the quartic case), but (2) we will need one more level of commutators for the additional root appearing in R_3.

- In general, for $n = 5$, we have $[(ijk), (k\ell m)] = (jkm)$.

- Thus, any cycle (jkm) can be written as a commutator of two other cycles, namely $[(ijk),(k\ell m)]$.

- But notice that this is true for any cycle (jkm), including (ijk) and $(k\ell m)$ on the left-hand side of the equality: $[(ijk),(k\ell m)] = (jkm)$. In other words, this formula can be applied to itself.

- Hence the cycle (jkm) can be written as a nested commutator of commutators as many as times as we want.

- Since a number $m \in \mathbb{N}$ of commutators allows us to discard precisely m levels of roots in a formula, we can actually discard any number of roots in any proposed formula for the quintic!

A short summary

Let $\mathscr{C}$ denote the space of coefficients of the quintic minus those leading to double roots. Let $\mathscr{S}$ denote the space of solutions to a quintic consisting of five distinct unordered complex numbers $\{s_1, \ldots, s_5\}$. Order these, in anyway you like when a fixed but arbitrary quintic is chosen. Suppose that a solution of the quintic equation can be expressed by a multi-valued function F.

- There is a onto map from the space of loops to the permutation group S_5.
- Suppose that $\gamma \in \pi_1(\mathscr{C})$ induces a cycle (123). Then picking a fixed branch of F, claim that $F \circ \gamma(0) = \gamma_1 = F \circ \gamma(1)$, which is a contradiction!
- The claim is easily verified by checking that (123) is a commutator of commutators in the permutation group S_5 on 5 symbols.

Paul Ramond, The Abel–Ruffini's Theorem: Complex but Not Complicated!, The American Mathematical Monthly, 129 (2022), 231-245.

LEO GOLDMAKHER, Arnold's elementary proof of the insolvabilty of the quintic

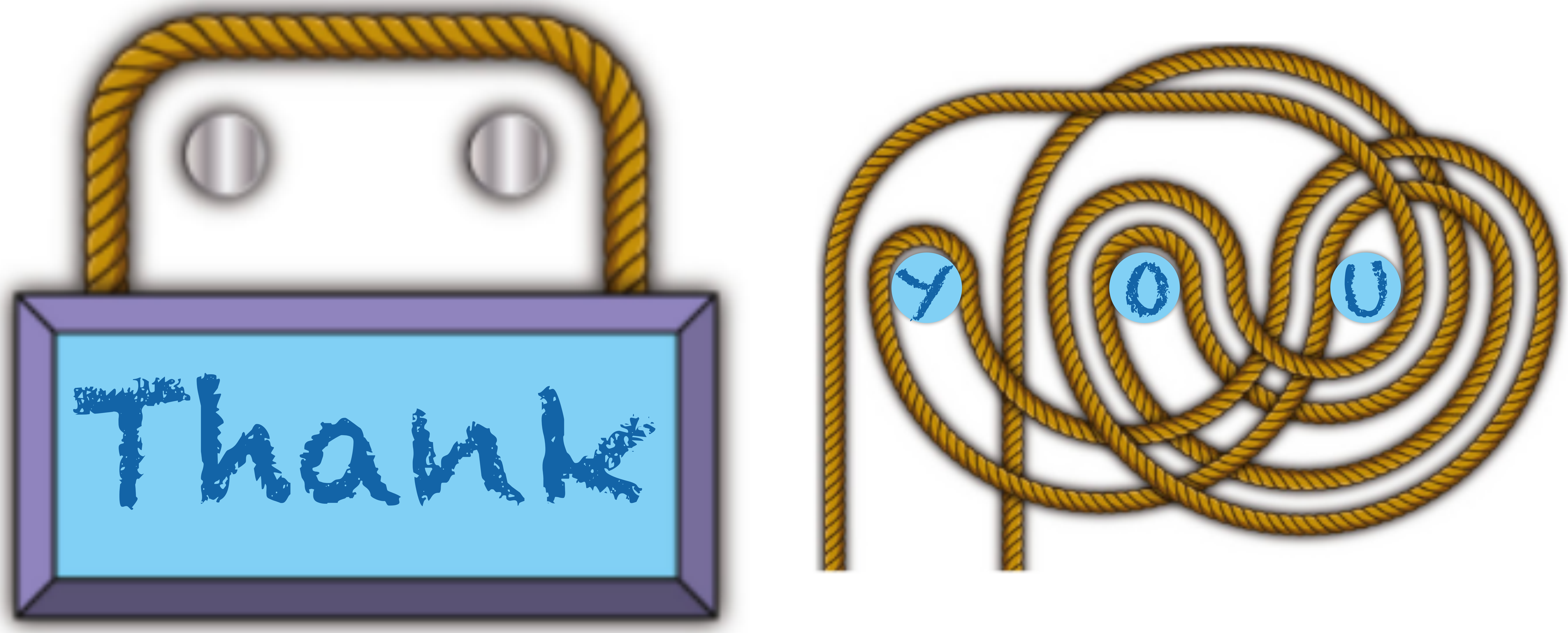
F. Akalin, Why is the Quintic Unsolvable? — akalin.com/quintic-unsolvabilityhttps://www.youtube.com/watch?v=BSHv9Elk1MU

Boaz Katz, https://www.youtube.com/watch?v=zeRXVL6qPk4&t=530s

Leo Stein, https://duetosymmetry.com/tool/polynomial-roots-toy/

Ramaprasad Saptharisi, https://www.youtube.com/watch?v=O5eH3x3sTNA

Carl Turner(Not all wrong), https://www.youtube.com/watch?v=BSHv9Elk1MU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